# 12. Kvadratické rovnice s komplexními koeficienty

KVADRATICKÁ ROVNICE

$ax^2 + bx + c = 0$

$a, b, c \in \mathbb{C}$ (koeficienty komplexní čísla)

– lze řešit různě (podle vhodnosti v daném příkladě)

## Příklady

① Řešte v C

a) $x^2 - 4ix - 8 = 0$

$(a=1,\ b=-4i,\ c=-8)$

$x_{1,2} = \dfrac{+4i \pm \sqrt{(-4i)^2 + 4.1.8}}{2}$

$x_{1,2} = \dfrac{4i \pm \sqrt{16i^2 + 32}}{2}$

$x_{1,2} = \dfrac{4i \pm \sqrt{-16+32}}{2} = \dfrac{4i \pm \sqrt{16}}{2}$

$x_{1,2} = \dfrac{4i \pm 4}{2} = \dfrac{2(2i \pm 2)}{2}$

$x_{1,2} = 2i \pm 2 \begin{cases} 2i + 2 = 2 + 2i \\ 2i - 2 = -2 + 2i \end{cases}$

$K = \{2+2i;\ -2+2i\}$

b) $x^2 - 6ix - 12 = 0$

DÚ $(a=1,\ b=-6i;\ c=-12)$

$x_{1,2} = \dfrac{6i \pm \sqrt{36i^2 + 4.12}}{2}$

$x_{1,2} = \dfrac{6i \pm \sqrt{-36+48}}{2}$

$x_{1,2} = \dfrac{6i \pm \sqrt{\overset{3.4}{12}}}{2} = \dfrac{6i \pm 2\sqrt{3}}{2}$

$x_{1,2} = \dfrac{2(3i \pm \sqrt{3})}{2} = 3i \pm \sqrt{3}$

$x_1 = 3i + \sqrt{3} = \sqrt{3} + 3i$

$x_2 = 3i - \sqrt{3} = -\sqrt{3} + 3i$

$K = \{\sqrt{3}+3i;\ -\sqrt{3}+3i\}$

c) $x^2 - 4ix - 3 = 0$

$(a=1,\ b=-4i;\ c=-3)$

$x_{1,2} = \dfrac{4i \pm \sqrt{16i^2 + 4.3}}{2} = \dfrac{4i \pm \sqrt{-4}}{2}$

$x_{1,2} = \dfrac{4i \pm \sqrt{4i^2}}{2} = \dfrac{4i \pm 2i}{2} = \dfrac{2(2i \pm i)}{2}$

$x_{1,2} = 2i \pm i = \begin{cases} 2i + i = 3i \\ 2i - i = i \end{cases}$

$K = \{3i;\ i\}$

d) $x^2 + 2x - ix + 3 - i = 0$

DÚ $x^2 + x(2-i) + 3 - i = 0$

$(a=1,\ b=2-i,\ c=3-i)$

$x_{1,2} = \dfrac{-(2-i) \pm \sqrt{(2-i)^2 - 4(3-i)}}{2}$

$x_{1,2} = \dfrac{-2+i \pm \sqrt{4 \cancel{-4i} + \overset{-1}{i^2} - 12 \cancel{+4i}}}{2}$

$x_{1,2} = \dfrac{-2+i \pm \sqrt{-9}}{2} = \dfrac{-2+i \pm \sqrt{9i^2}}{2}$

$x_{1,2} = \dfrac{-2+i \pm 3i}{2} = \begin{cases} \dfrac{-2+i+3i}{2} = \dfrac{-2+4i}{2} = -1+2i \\ \dfrac{-2+i-3i}{2} = \dfrac{-2-2i}{2} = -1-i \end{cases}$

$K = \{-1+2i,\ -1-i\}$

② Řešte v C

a) $x^2 - 2x - ix - 1 + 7i = 0$

$x^2 - (2+i)x - 1 + 7i = 0$

$(a=1,\ b=-(2+i),\ c=-1+7i)$

$x_{1,2} = \dfrac{2+i \pm \sqrt{[-(2+i)]^2 - 4.(-1+7i)}}{2} = \dfrac{2+i \pm \sqrt{\overset{(a+b)^2}{(2+i)^2} + 4 - 28i}}{2} = \dfrac{2+i \pm \sqrt{\overset{a^2+2ab+b^2}{4+4i+i^2} + 4 - 28i}}{2}$

(*) $x_{1,2} = \dfrac{2+i \pm \sqrt{7-24i}}{2}$ PROBLÉM – URČIT ODMOCNINU $\sqrt{7-24i}$

– po určení odmocniny dosadíme do této rovnice zpět

– způsoby různé

(1. způs.) ALGEBRAICKY (ZKUSMO)!

$$\sqrt{7-24i} = a+bi \quad /^2 \quad a,b \in R$$

$$7-24i = a^2 + 2abi + (bi)^2$$

$$7-24i = a^2 + 2abi + \underbrace{b^2 i^2}_{-b^2}$$

$$\underbrace{7}_{Re} - \underbrace{24i}_{Im} = \underbrace{a^2 - b^2}_{Re} + \underbrace{2abi}_{Im}$$

z rovnosti kompl. č. (porovn. Re, Im)

$$\left.\begin{aligned} 7 &= a^2 - b^2 \\ -24 &= 2ab \end{aligned}\right\} \text{soustava}$$

$$\left.\begin{aligned} a^2 - b^2 &= 7 \\ ab &= -12 \end{aligned}\right\} \text{ZKUSMO (nevím-li} \Rightarrow \text{2. způs.)} \; (**)$$

– hledám čísla, jejichž součin je −12 a rozdíl druhých mocnin 7

$-12:$ $-12, 1$ → $144-1 \neq 7$ ← $12, -1$; $-6, 2$ → $36-4 \neq 7$ ← $6, -2$; $-4, 3$ → $16-9 = 7$ ← $4, -3$

$\Rightarrow a_1 = -4, b_1 = 3$

$\Rightarrow a_2 = 4, b_2 = 3$

$a_1 = 4 \qquad a_2 = -4$

$b_1 = -3 \qquad b_2 = 3$

$4-3i \qquad -4+3i$

$$\sqrt{7-24i} = a+bi = \left\langle \begin{matrix} 4-3i \\ -4+3i \end{matrix} \right. \rightarrow \text{nemusím}$$

– dos. do (*) [viz dříve] – STAČÍ JEDNA ODMOCNINA (rychlejší) – z druhé dostaneme stejné kořeny

$$x_{1,2} = \frac{2+i \pm \sqrt{7-24i}}{2} = \frac{2+i \pm (4-3i)}{2} = \left\langle \begin{matrix} \dfrac{2+i+4-3i}{2} = \dfrac{6-2i}{2} = 3-i \\[2ex] \dfrac{2+i-(4-3i)}{2} = \dfrac{2+i-4+3i}{2} = \dfrac{-2+4i}{2} = -1+2i \end{matrix} \right.$$

$$K = \{3-i, -1+2i\}$$

(2. způs.) ALGEBRAICKY POČETNĚ (NEJDE-LI ZKUSMO)

$$(**) \left\{ \begin{aligned} a^2 - b^2 &= 7 \\ ab &= -12 \Rightarrow a = \frac{-12}{b} \end{aligned} \right.$$

$$\left(-\frac{12}{b}\right)^2 - b^2 = 7$$

$$\frac{144}{b^2} - b^2 = 7 \quad / \cdot b^2$$

$$144 - b^4 = 7b^2$$

$$0 = b^4 + 7b^2 - 144$$

rce 4. stupně, vhodná subst.

subst. $y = b^2$ => rce má tvar

$$y^2 + 7y - 144 = 0 \quad \text{(kv. r.)}$$

$$y_{1,2} = \frac{-7 \pm \sqrt{49 + 4 \cdot 144}}{2} = \frac{-7 \pm \sqrt{625}}{2}$$

$$y_{1,2} = \frac{-7 \pm 25}{2} = \left\langle \begin{matrix} \dfrac{-7+25}{2} = 9 = y_1 \\[2ex] \dfrac{-7-25}{2} = -16 = y_2 \end{matrix} \right.$$

zpět k subst. $(y = b^2)$

$9 = b^2 \qquad -16 = b^2$

$|b| = 3 \qquad$ nemá v R řešení $(a,b \in R)$

$b_1 = 3 \qquad b_2 = -3$

$a_1 = -\frac{12}{3} \qquad a_2 = \frac{-12}{-3}$

$a_1 = -4 \qquad a_2 = 4$

$$\sqrt{7-24i} = a+bi \quad (a,b \in R)$$

$$\sqrt{7-24i} = \left\langle \begin{matrix} -4+3i \\ 4-3i \end{matrix} \right. \quad \text{(viz 1. způs.) dos. do (*)}$$

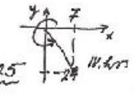
(3. způs.) GONIOMETRICKY (S KALKULAČKOU)

$$x = \sqrt{7-24i}$$

$a = 7-24i$

na gon. tvar

$|a| = \sqrt{49+24^2} = \sqrt{625} = 25$




$\sin \alpha = \frac{-24}{25} \Rightarrow \alpha' = 73{,}74°$

IV.kv.

$\alpha = 360° - 73{,}74°$

$\alpha = 286{,}25°$

$$x_k = \sqrt{|a|}\left(\cos\frac{\alpha + k \cdot 360°}{2} + i \sin\frac{\alpha + k \cdot 360°}{2}\right)$$

$$x_0 = \sqrt{25}\left(\cos\frac{286{,}25°}{2} + i \sin\frac{286{,}25}{2}\right) =$$

$$x_0 = 5(-0{,}8 + 0{,}6i) = -4+3i$$

$$x_1 = 5\left[\cos\left(\frac{286{,}25°}{2} + 180°\right) + i \sin\left(\frac{286{,}25}{2} + 180°\right)\right]$$

$$x_1 = 5(0{,}8 - 0{,}6i) = 4-3i$$

(4. způs.) GONIOMETRICKY BEZ KALKULAČKY S VYUŽITÍM GONIOMETRICKÝCH VZORCŮ – určení $\cos\frac{\alpha}{2}$, $\sin\frac{\alpha}{2}$

(5. způs.) S VYUŽITÍM VZORCE PRO KOŘENY

- kvadr. rovnice $ax^2+bx+c=0$ s komplexními koeficienty má v oboru KOMPLEXNÍCH čísel PRÁVĚ 2 KOŘENY $\quad D=b^2-4ac$

pro $D=0$ $\quad x_{1/2}=\frac{-b}{2a}$ (splynou)

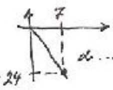
pro $D\neq 0$ $\quad x_{1/2}=\dfrac{-b\pm\sqrt{|D|}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)}{2a}$ $\qquad$ $\alpha$… argument (úhel) D vyj. v goniom. tvaru

$$x^2-2x-ix-1+7i=0$$
$$x^2-(2+i)x-1+7i=0$$

$(a=1,\ b=-(2+i),\ c=-1+7i)$

$$D=b^2-4ac=\left[-(2+i)\right]^2-4.(-1+7i)\,]=(2+i)^2-4-28i=4+4i+\underbrace{i^2}_{-1}+4-28i=\underbrace{7-24i}_{a+bi}\ (=D)$$

$$|D|=\sqrt{7^2+(-24)^2}=\sqrt{49+24^2}=\sqrt{625}=25$$




$$\cos\alpha=\frac{a}{|D|}=\frac{7}{25}\ \Rightarrow\left|\cos\frac{\alpha}{2}\right|=\sqrt{\frac{1+\cos\alpha}{2}}$$

IV.kv. $\alpha$ => $\frac{\alpha}{2}$ II.kv. $\quad$ => II.kv. (−)

$$\sin\alpha=\frac{b}{|D|}=\frac{-24}{25}\ \Rightarrow\left|\sin\frac{\alpha}{2}\right|=\sqrt{\frac{1-\cos\alpha}{2}}$$

IV.kv. $\quad$ => II.kv. (+)

nepotřebujeme $\alpha$, ale $\frac{\alpha}{2}$ - využ. vzorců pro poloviční úhel

POZOR NA ZNAMÉNKA

$$\left|\cos\frac{\alpha}{2}\right|=\sqrt{\frac{1+\cos\alpha}{2}}=\sqrt{\frac{1+\frac{7}{25}}{2}}=\sqrt{\frac{\frac{25+7}{25}}{2}}=\sqrt{\frac{\frac{32}{25}}{2}}=\sqrt{\frac{32}{2.25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}\ \Rightarrow\ \cos\frac{\alpha}{2}=-\frac{4}{5}$$

II.kv. (−)

$$\left|\sin\frac{\alpha}{2}\right|=\sqrt{\frac{1-\cos\alpha}{2}}=\sqrt{\frac{1-\frac{7}{25}}{2}}=\sqrt{\frac{\frac{25-7}{25}}{2}}=\sqrt{\frac{\frac{18}{25}}{2}}=\sqrt{\frac{18}{2.25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}\ \Rightarrow\ \sin\frac{\alpha}{2}=\frac{3}{5}$$

II.kv. (+)

$$x_{1/2}=-b\pm\sqrt{|D|}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$$

$$x_{1/2}=\frac{2+i\pm\sqrt{25}\left(-\frac{4}{5}+\frac{3}{5}i\right)}{2}=\frac{2+i\pm5\left(-\frac{4}{5}+\frac{3}{5}i\right)}{2}=\begin{cases}\dfrac{2+i+(-4+3i)}{2}=\dfrac{-2+4i}{2}=\underline{-1+2i}\\[2mm]\dfrac{2+i-(-4+3i)}{2}=\dfrac{6-2i}{2}=\underline{3-i}\end{cases}$$

VSUVKA

- ODVOZENÍ VZTAHŮ $\left|\sin\frac{x}{2}\right|=\sqrt{\frac{1-\cos x}{2}}$ $\quad\left|\cos\frac{x}{2}\right|=\sqrt{\frac{1+\cos x}{2}}$

- vyjdeme ze vzorce: $\sin^2x+\cos^2x=1\ \Rightarrow\ \sin^2\alpha=1-\cos^2\alpha,\ \cos^2\alpha=1-\sin^2\alpha$

OBDOBNÉ

| | |
|---|---|
| $\cos2\alpha=\cos^2\alpha-\sin^2\alpha$ | $\cos2\alpha=\cos^2\alpha-\sin^2\alpha$ |
| $\cos2\alpha=\cos^2\alpha-(1-\cos^2\alpha)$ | $\cos2\alpha=1-\sin^2\alpha-\sin^2\alpha$ |
| $\cos2\alpha=2\cos^2\alpha-1$ | $\cos2\alpha=1-2\sin^2\alpha$ |
| $2\cos^2\alpha=1+\cos2\alpha$ | $2\sin^2\alpha=1-\cos2\alpha$ |
| $\cos^2\alpha=\frac{1+\cos2\alpha}{2}$ | $\sin^2\alpha=\frac{1-\cos2\alpha}{2}$ |
| $\lvert\cos\alpha\rvert=\sqrt{\frac{1+\cos2\alpha}{2}}$ | $\lvert\sin\alpha\rvert=\sqrt{\frac{1-\cos2\alpha}{2}}$ |
| ozn. $\alpha=\frac{x}{2}\Rightarrow2\alpha=x$ | ozn. $\alpha=\frac{x}{2}\Rightarrow2\alpha=x$ |
| $\left\lvert\cos\frac{x}{2}\right\rvert=\sqrt{\frac{1+\cos x}{2}}$ | $\lvert\sin\alpha\rvert=\sqrt{\frac{1-\cos x}{2}}$ |

[ dále pak $\left|\mathrm{tg}\frac{x}{2}\right|=\sqrt{\frac{1-\cos x}{1+\cos x}}$ $\quad\left|\mathrm{cotg}\frac{x}{2}\right|=\sqrt{\frac{1+\cos x}{1-\cos x}}$ $\quad$ DALŠÍ UŽITEČNÉ VZORCE ]

$[\mathrm{tg}\,x=\frac{\sin x}{\cos x}]$ $\qquad$ $[\mathrm{cotg}\,x=\frac{\cos x}{\sin x}]$

b) $x^2+3x+10i=0 \qquad O=\mathbb{C} \quad D=\mathbb{C}$

ŘD $(a=1,\ b=3,\ c=10i)$

$$x_{1,2}=\frac{-3\pm\sqrt{9-4\cdot 10i}}{2}=\frac{-3\pm\sqrt{9-40i}}{2} \quad (*)$$

$\sqrt{9-40i}=a+bi \quad /^2$ umocníme P stranu jako dvojčlen $(A+B)^2=A^2+2AB+B^2$

$9-40i=a^2+2abi+(bi)^2$

$9-40i=a^2+2abi+b^2\underbrace{i^2}_{-1}$

$\underbrace{9}_{Re}\underbrace{-40i}_{Im}=\underbrace{a^2-b^2}_{Re}+\underbrace{2abi}_{Im}$

z rovnosti kompl. č. (porovnáme Re, Im)

$9=a^2-b^2$

$-40=2ab \quad /:2$

$a^2-b^2=9$

$ab=-20 \Rightarrow$ ZKUSMO $5,-4 \quad 25-16=9$; $-5,4 \quad 25-16=9$

hledáme cel. čísla $a, b$

$a_1=5 \quad a_2=-5$

$b_1=-4 \quad b_2=4$

tedy $\sqrt{9-40i}=\begin{cases}5-4i\\-5+4i\end{cases}$

dos. do (*) (stačí jedna z odmocnin)

$$x_{1,2}=\frac{-3\pm(5-4i)}{2}=\begin{cases}\frac{-3+(5-4i)}{2}=\frac{2-4i}{2}=1-2i\\[4pt]\frac{-3-(5-4i)}{2}=\frac{-8+4i}{2}=-4+2i\end{cases}$$

$K=\{1-2i,\ -4+2i\}$

c) $x^2-2x+9+6i=0 \qquad O=\mathbb{C} \quad D=\mathbb{C}$

ŘD $(a=1,\ b=-2,\ c=9+6i)$

$$x_{1,2}=\frac{2\pm\sqrt{4-4(9+6i)}}{2}=\frac{2\pm\sqrt{4-36-24i}}{2}=$$

$$=\frac{2\pm\sqrt{-32-24i}}{2}=\frac{2\pm\sqrt{4(-8-6i)}}{2}$$

$$=\frac{2\pm2\sqrt{-8-6i}}{2}=1\pm\sqrt{-8-6i} \quad (*)$$

$\sqrt{-8-6i}=a+bi$

$-8-6i=a^2+2abi+b^2\underbrace{i^2}_{-1}$

$\underbrace{-8}_{Re}\underbrace{-6i}_{Im}=\underbrace{a^2-b^2}_{Re}+\underbrace{2abi}_{Im}$

$-8=a^2-b^2$

$-6=2ab \quad /:2$

$a^2-b^2=-8$

$ab=-3 \rightarrow$ ZKUSMO $3,-1 \quad 9-1\neq-8$; $-1,3 \quad 1-9=-8$ pl.; $1,-3 \quad 1-9=-8$ pl.

$a_1=-1 \quad a_2=1$

$b_1=3 \quad b_2=-3$

$\sqrt{-8-6i}\begin{cases}-1+3i\\1-3i\end{cases}$ dos. do (*)

$$x_{1,2}=1\pm(1-3i)=\begin{cases}1+(1-3i)=2-3i\\1-(1-3i)=3i\end{cases}$$

d) $x(3-x)=3-i \qquad O=\mathbb{C} \quad D=\mathbb{C}$

ŘD $3x-x^2=3-i$

$0=x^2-3x+3-i$

$(a=1,\ b=-3,\ c=3-i)$

$$x_{1,2}=\frac{+3\pm\sqrt{9-4(3-i)}}{2}=\frac{3\pm\sqrt{-3+4i}}{2} \quad (*)$$

$\sqrt{-3+4i}=a+bi$

$-3+4i=a^2+2abi+(bi)^2 \Rightarrow [b^2\underbrace{i^2}_{-1}=-b^2]$

$\underbrace{-3}_{Re}+\underbrace{4i}_{Im}=\underbrace{a^2-b^2}_{Re}+\underbrace{2abi}_{Im}$

$a^2-b^2=-3$

$2ab=4$

$a^2-b^2=-3$

$ab=2$ ZKUSMO $1,2 \quad 1-4=-3$ pl. $(-1,-2$ pl.$)$; $2,1 \quad 4-1=3$ nepl.

$a_1=1 \quad a_2=-1$

$b_1=2 \quad b_2=-2$

$\sqrt{-3+4i}=\begin{cases}1+2i\\-1-2i\end{cases}$ [dos. do (*) stačí 1 z odm.]

$$x_{1,2}=\frac{3\pm(1+2i)}{2}=\begin{cases}\frac{3+(1+2i)}{2}=\frac{4+2i}{2}=2+i\\[4pt]\frac{3-(1+2i)}{2}=\frac{2-2i}{2}=1-i\end{cases}$$

e) $x^2+2x-3xi-5-5i=0 \qquad O=\mathbb{C} \quad D=\mathbb{C}$

ŘD $x^2+x(2-3i)-5(1+i)=0$

$(a=1,\ b=2-3i;\ c=-5(1+i))$

$$x_{1,2}=\frac{-2+3i\pm\sqrt{(2-3i)^2+4\cdot5(1+i)}}{2}=$$

$$=\frac{3i-2\pm\sqrt{4-12i+9\underbrace{i^2}_{-1}+20+20i}}{2}=$$

$$=\frac{3i-2\pm\sqrt{15+8i}}{2} \quad (*)$$

$\sqrt{15+8i}=a+bi$

$15+8i=a^2+2abi+\underbrace{b^2i^2}_{-b^2}$

$a^2-b^2=15$

$2ab=8$

$a^2-b^2=15$

$ab=4$ $\quad 4,1 \quad 16-1=15$; $-4,-1 \quad 16-1=15$

$a_1=4 \quad a_2=-4$

$b_1=1 \quad b_2=-1$

$\sqrt{15+8i}=\begin{cases}4+i\\-4-i\end{cases}$ dos. do (*)

$$x_{1,2}=\frac{3i-2\pm(4+i)}{2}=\begin{cases}\frac{3i-2+4+i}{2}=\frac{2+4i}{2}=1+2i\\[4pt]\frac{3i-2-(4+i)}{2}=\frac{2i-6}{2}=-3+i\end{cases}$$